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Reversible extensions of irreversible dynamical systems: the C^*-method

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Abstract. A construction of a reversible extension of irreversible dynamical systems is presented. It is based on calculating the maximal ideal spaces of the C^*-algebras generated by these systems and the corresponding reversible extensions of endomorphisms. Connections between the objects that arise and dynamical systems of Smale horseshoe and other types are revealed.

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§ 1. Introduction. Extensions of C^*-algebras by partial isometries, extensions of dynamical systems and coefficient algebras

The problems at the heart of this paper arise in a variety of areas of analysis, including the theory of C^*-algebras associated with automorphisms and endomorphisms and, in particular, the theory of crossed products, the theory of dynamical systems and their inverse (projective) limits, spectral analysis of weighted shift operators and transfer operators.

To explain our motivation we first give some simple examples, as models illustrating the statement of the problem and the C^*-algebraic and dynamical objects arising in its analysis.

1.1. Extending C^*-algebras by partial isometries. Extending endomorphisms to automorphisms.

Example 1.1 (the Töplitz algebra). Recall the construction of the classical Töplitz algebra. Let $H = l^2(\mathbb{N})$ and let $\mathscr{A} \subset L(H)$ be the C^*-algebra of operators of multiplication by bounded convergent sequences

$$a = (a(k)) \in l^\infty(\mathbb{N}), \qquad \lim_{k\to+\infty} a(k) = a(+\infty).$$

We define an isometric operator $U \in L(H)$ (a one-sided shift) by the formula

$$(Uh)(k) = \begin{cases} 0, & k = 0, \\ h(k-1), & k > 0, \end{cases} \qquad h \in H.$$

The *Töplitz algebra* is the C^*-algebra $C^*(\mathscr{A}, U)$ generated by $\mathscr{A}$ and the operator U.

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After some straightforward calculations, we see that the maps

$$\mathscr{A} \ni a \mapsto \delta(a) = UaU^*,$$
$$\mathscr{A} \ni a \mapsto \delta_*(a) = U^*aU$$

are endomorphisms of $\mathscr{A}$:

$$\delta(a)(k) = \begin{cases} 0, & k = 0, \\ a(k-1), & k > 0, \end{cases} \tag{1}$$
$$\delta_*(a)(k) = a(k+1), \qquad a(\,\cdot\,) \in \mathscr{A}.$$

The Töplitz algebra is a classical object of analysis, with plenty of applications. One might say it has the 'deficiency' that it is naturally associated with an endomorphism δ, rather than with an automorphism. Can this be corrected? In the next example we show how to do this in a simple way.

Example 1.2 (the extended Töplitz algebra). Let $H = l^2(\mathbb{Z})$ and let $\mathscr{A} \subset L(H)$ be the C^*-algebra of operators of multiplication by bounded sequences vanishing on the negative part of $\mathbb{Z}$ and with limits at $+\infty$:

$$a = (a(k)) \in l^\infty(\mathbb{Z}), \qquad \forall_{k<0}\, a(k) = 0, \qquad \lim_{k\to+\infty} a(k) = a(+\infty).$$

This algebra $\mathscr{A}$ is clearly isomorphic (as a C^*-algebra) to the algebra $\mathscr{A}$ from the previous example.

Let $U \in L(H)$ be the unitary operator (a two-sided shift) defined by the formula

$$(Uh)(k) = h(k-1), \qquad h \in H.$$

Note that in this case the map $\mathscr{A} \ni a \mapsto \delta(a) = UaU^*$ is an endomorphism of the algebra $\mathscr{A}$:

$$\delta(a)(k) = \begin{cases} 0, & k \leqslant 0, \\ a(k-1), & k > 0. \end{cases}$$

However, the map $\mathscr{A} \ni a \mapsto \delta_*(a) = U^*aU$ is no longer an endomorphism of $\mathscr{A}$ because $U^*\mathscr{A}U \nsubseteq \mathscr{A}$.

Consider the algebra $\mathscr{B} \subset L(H)$ of operators of multiplication by bounded sequences with limits at $\pm\infty$:

$$b = (b(k)) \in l^\infty(\mathbb{Z}), \qquad \lim_{k\to\pm\infty} b(k) = b(\pm\infty).$$

We readily see that

$$C^*(\mathscr{A}, U) = C^*(\mathscr{B}, U).$$

Moreover, the maps $\delta(\,\cdot\,) = U(\,\cdot\,)U^*$ and $\delta_*(\,\cdot\,) = U_*(\,\cdot\,)U$ are now automorphisms of the algebra $\mathscr{B}$:

$$\delta(b)(k) = b(k-1), \qquad \delta_*(b)(k) = (k+1). \tag{2}$$

We shall call $C^*(\mathscr{B}, U)$ the *extended Töplitz algebra*.

It is well known that

$$C^*(\mathscr{B}, U) \cong \mathscr{B} \times_\delta \mathbb{Z},$$

where on the right-hand side we have the crossed product of the algebra $\mathscr{B}$ and the group $\mathbb{Z}$ acting on $\mathscr{B}$ by the automorphisms δ^n, $n \in \mathbb{Z}$. Here the elements of the crossed product can be represented by formal series with coefficients in $\mathscr{B}$.

In this example it is natural to regard $\mathscr{B}$ as an extension of the algebra $\mathscr{A}$ and $C^*(\mathscr{B}, U)$ as an extension of the Töplitz algebra $C^*(\mathscr{A}, U)$ from Example 1.1; then the endomorphism $\delta\colon \mathscr{A} \to \mathscr{A}$ described in Example 1.1 by formula (1) extends to the automorphism $\delta\colon \mathscr{B} \to \mathscr{B}$ described here by formula (2).

Thus, by extending the Töplitz algebra (associated with an endomorphism) we arrive at the construction of a crossed product associated with an automorphism.

The important role of structures similar to Töplitz algebras and crossed products is well-known in analysis.

Starting from this example we arrive in a natural way at the following problem: can we carry out a construction similar to that in these examples in a more general setting, that is, can we describe a general construction of an extension $\mathscr{B}$ of an algebra $\mathscr{A}$ such that the extension to $\mathscr{B}$ of a fixed endomorphism $\delta\colon \mathscr{A} \to \mathscr{A}$ is an automorphism? In this paper we present a complete answer to this question in the case of a commutative algebra $\mathscr{A}$.

In reality, our question is closely connected with the problem of constructing reversible extensions of irreversible dynamical systems. To illustrate this connection we will look at the above examples from the point of view of topological dynamics.

1.2. Reversible extensions of irreversible dynamical systems.

Example 1.3 (a dynamical interpretation of Example 1.2). For technical reasons and for convenience, alongside the objects from Example 1.2, we shall also consider the C^*-algebra $\overline{\mathscr{A}} := C^*(\mathscr{A}, 1)$ generated by the algebra $\mathscr{A}$ in Example 1.2 and the identity operator 1. Obviously, $\overline{\mathscr{A}}$ is the algebra of operators $a \in \mathscr{B}$ of multiplication by sequences that are constant for $n < 0$. In essence, the observations made in Example 1.2 do not change, since $C^*(\overline{\mathscr{A}}, U) = C^*(\mathscr{B}, U)$ and the algebra $\mathscr{B}$ is an extension of $\overline{\mathscr{A}}$; furthermore, the map $\delta(\,\cdot\,) = U(\,\cdot\,)U^*$ is an endomorphism of the algebra $\overline{\mathscr{A}}$ and $U^*\overline{\mathscr{A}}U \nsubseteq \mathscr{A}$.

Note that

$$\overline{\mathscr{A}} \cong C(\overline{\mathbb{N}}) \quad \text{and} \quad \mathscr{B} \cong C(\overline{\mathbb{Z}}),$$

where $\overline{\mathbb{N}} = \mathbb{N} \cup \{-\infty, +\infty\}$ and $\overline{\mathbb{Z}} = \mathbb{Z} \cup \{-\infty, +\infty\}$.

Consider the map $\alpha\colon \overline{\mathbb{N}} \to \overline{\mathbb{N}}$ defined by the formulae

$$\alpha(k) = k - 1, \quad k > 0, \qquad \alpha(0) = -\infty, \qquad \alpha(\pm\infty) = \pm\infty,$$

and the map (the homeomorphism) $\widetilde{\alpha}\colon \overline{\mathbb{Z}} \to \overline{\mathbb{Z}}$ defined by the formulae

$$\widetilde{\alpha}(k) = k - 1, \quad k \in \mathbb{Z}, \qquad \alpha(\pm\infty) = \pm\infty.$$

The map α defines an endomorphism δ of the algebra $\overline{\mathscr{A}}$:

$$\delta(a) = a \circ \alpha, \qquad a \in \overline{\mathscr{A}},$$

while $\widetilde{\alpha}$ defines an automorphism δ (an extension of the above endomorphism) of $\mathscr{B}$:

$$\delta(b) = b \circ \widetilde{\alpha}, \qquad b \in \mathscr{B}.$$

Consider also the map $\Psi\colon \overline{\mathbb{Z}} \to \overline{\mathbb{N}}$ defined by the formulae

$$\Psi(n) = n, \quad n \geqslant 0, \qquad \Psi(n) = -\infty, \quad n < 0, \qquad \Psi(\pm\infty) = \pm\infty.$$

It establishes a semiconjugacy between the dynamical systems $(\overline{\mathbb{Z}}, \widetilde{\alpha})$ and $(\overline{\mathbb{N}}, \alpha)$ in the following sense: $\alpha(\Psi(x)) = \Psi(\widetilde{\alpha}(x))$, $x \in \overline{\mathbb{Z}}$, and since Ψ is surjective, the dynamical system $(\overline{\mathbb{Z}}, \widetilde{\alpha})$ can naturally be regarded as a reversible extension of the irreversible dynamical system $(\overline{\mathbb{N}}, \alpha)$.

We see that associated with the extension of the endomorphism δ from the algebra $\overline{\mathscr{A}}$ to an automorphism of the algebra $\mathscr{B} \supset \overline{\mathscr{A}}$ is an extension of the irreversible dynamical system $(\overline{\mathbb{N}}, \alpha)$ to the reversible dynamical system $(\overline{\mathbb{Z}}, \widetilde{\alpha})$.

Remark 1.4. As concerns the map Ψ, it has the following important property which is perhaps worth a mention. Identifying the sets $\overline{\mathbb{N}}$ and $\overline{\mathbb{Z}}$ with the sets of multiplicative functionals on the algebras $\overline{\mathscr{A}}$ and $\mathscr{B}$, respectively, we note that

$$\Gamma := \{-\infty, n < 0\} = \Psi^{-1}(-\infty) \subset \overline{\mathbb{Z}}$$

coincides with the set of multiplicative functionals on $\mathscr{B}$ extending the multiplicative functional $-\infty \in \overline{\mathbb{N}}$ on $\mathscr{A}$ (cf. the general case described in § 2.2 below and also in § 4, where a similar map is defined by formula (68)).

As we shall show in Theorem 2.2, there is a general relation between algebra endomorphisms and dynamical systems, therefore the above example and its C^*-algebraic description lead to the following question: what form should a general construction of an invertible extension of a non-invertible dynamical system take, and how is this construction connected with the construction of an extension of a C^*-algebra and an extension of an endomorphism to an automorphism? We will also answer this question in our paper.

In fact, all the above questions and the objects described above come together in the construction of the so-called coefficient algebra, which we will now discuss.

1.3. Constructing the coefficient algebra. The concept of a coefficient algebra was introduced in [1] in connection with the investigation of extensions of C^*-algebras by partial isometries. More precisely, in [1] the authors considered the following object. Let H be a Hilbert space and $\mathscr{A} \subset L(H)$ be a *-algebra containing the unity 1 of the algebra $L(H)$. The paper aimed to describe the C^*-extensions of the algebra $\mathscr{A}$ associated with the maps

$$\delta(x) = UxU^*, \qquad \delta_*(x) = U^*xU, \qquad x \in L(H), \tag{3}$$

where $U \in L(H)$, $U \neq 0$. It is clear that δ and δ_* are continuous linear transformations of $L(H)$ ($\|\delta\| = \|\delta_*\| = \|U^2\|$) and $\delta(x^*) = \delta(x)^*$, $\delta_*(x^*) = \delta_*(x)^*$. Taking their powers δ^k and δ_*^k, $k = 0, 1, 2, \dots$, we shall set for convenience $\delta^0(x) = \delta_*^0(x) = x$.

Note that if $\delta\colon \mathscr{A} \to L(H)$ is a morphism, then

$$UU^* = \delta(1) = \delta(1^2) = \delta^2(1) = (UU^*)^2$$

so that U is a partial isometry.

In [1] the authors considered the C^*-algebra $C^*(\mathscr{A}, U)$ generated by $\mathscr{A}$ and U under the additional assumption that $\mathscr{A}$ be the *coefficient algebra* of the algebra $C^*(\mathscr{A}, U)$; this meant that $\mathscr{A}$ had the following three properties:

$$\mathscr{A} \ni a \to \delta(a) = UaU^* \in \mathscr{A}, \tag{4}$$

$$\mathscr{A} \ni a \to \delta_*(a) = U^*aU \in \mathscr{A}, \tag{5}$$

$$Ua = \delta(a)U, \qquad a \in \mathscr{A}. \tag{6}$$

As shown in [1], algebras with properties (4)–(6) do indeed play the role of 'coefficients' of $C^*(\mathscr{A}, U)$; namely, it is proved in Proposition 2.4 of [1] that *if a* **-algebra* $\mathscr{A}$ *and* U *satisfy conditions* (4)–(6), *then the vector space of finite sums*

$$x = U^*a_{\overline{N}} + \cdots + U^*a_{\overline{1}} + a_0 + a_1U + \cdots + a_NU^N, \tag{7}$$

where a_k, $a_{\overline{k}} \in \mathscr{A}$ *and* $N \in \mathbb{N} \cup \{0\}$, *is a dense* **-subalgebra of the* C^**-algebra* $C^*(\mathscr{A}, U)$.

It is useful to observe that (6) also has another equivalent form, as stated by the following proposition.

Proposition 1.5 (see [1], Proposition 2.2). *Let* $\mathscr{A}$ *be a* C^**-subalgebra of* $L(H)$, *let* $1 \in \mathscr{A}$ *and* $U \in L(H)$. *Then the following conditions are equivalent*:

(i) $Ua = \delta(a)U$, $a \in \mathscr{A}$;

(ii) U *is a partial isometry and*

$$U^*U \in \mathscr{A}', \tag{8}$$

where $\mathscr{A}'$ *is the commutant of* $\mathscr{A}$;

(iii) $U^*U \in \mathscr{A}'$ *and* $\delta\colon \mathscr{A} \to \delta(\mathscr{A})$ *is a morphism.*

It follows from condition (6) that the map $\delta(a) = UaU^*$, $a \in \mathscr{A}$, is a morphism. Hence each coefficient algebra is associated with an endomorphism δ generated by a partial isometry U.

In [1] the authors explained how an algebra satisfying conditions (4)–(6) can be constructed from some initial algebra satisfying only some or even none of these conditions. Here we present the part of this construction we require, for the case of a commutative algebra $\mathscr{A}$.

Let

$$\overline{E_*(\mathscr{A})} = \overline{\left\{\bigcup_{n=0}^{\infty} \delta_*^n(\mathscr{A})\right\}} \tag{9}$$

be the C^*-algebra generated by $\bigcup_{n=0}^{\infty} \delta_*^n(\mathscr{A})$.

Remark 1.6. Recalling the algebras $\mathscr{B}$ and $\overline{\mathscr{A}}$ from Examples 1.2 and 1.3 above we note that

$$\mathscr{B} = \overline{E_*(\overline{\mathscr{A}})}.$$

For this reason the algebra $\overline{E_*(\mathscr{A})}$ in (9) with be of crucial importance for our analysis of the problem under consideration.

The following result was established in [1], Proposition 4.1.

Proposition 1.7. *Let $\mathscr{A}$ be a commutative C^*-subalgebra of $L(H)$ containing* 1. *Let δ be an endomorphism of $\mathscr{A}$ and let $U^*U \in \mathscr{A}'$. Then the C^*-algebra*

$$\overline{E_*(\mathscr{A})} = \overline{\left\{\bigcup_{n=0}^{\infty} \delta_*^n(\mathscr{A})\right\}}$$

is a minimal commutative coefficient algebra for $C^*(\mathscr{A}, U)$ and both

$$\delta \colon \overline{E_*(\mathscr{A})} \to \overline{E_*(\mathscr{A})} \quad \text{and} \quad \delta_* \colon \overline{E_*(\mathscr{A})} \to \overline{E_*(\mathscr{A})}$$

are endomorphisms.

Here $\delta \colon \overline{E_*(\mathscr{A})} \to \overline{E_*(\mathscr{A})}$ is the extension of the endomorphism $\delta \colon \mathscr{A} \to \mathscr{A}$ and δ_* plays the role of its inverse.

This result is the starting point for the C^*-algebraic analysis of the objects that are under consideration in this paper. In particular, in combination with the discussion above, it leads us to the natural problem of describing the maximal ideal space of the coefficient algebra $\overline{E_*(\mathscr{A})}$ in terms of the maximal ideal space of the algebra $\mathscr{A}$ and the action δ. To solve this problem is one of the central aims of this paper.

We point out that [2] contains several concrete examples of the maximal ideal space $\overline{E_*(\mathscr{A})}$ for $\mathscr{A} = C[a, b]$, when δ is generated by a continuous map with a special form

$$\alpha \colon [a, b] \to [a, b].$$

Remark 1.8. The role of coefficient algebras in the C^*-theory is fairly important since they are the major structural elements of crossed products associated with endomorphisms (see [3]–[5]). Hence the solution of the above-mentioned problems also allows us to construct the corresponding crossed products in a natural fashion (see [4]).

The paper is organized as follows. In § 2 we introduce the concepts and notation required for what follows and also present several facts (mostly known) about the structure of endomorphisms of commutative algebras and discuss their relations to dynamical systems. Our main result, the description of the maximal ideal space of the algebra $\overline{E_*(\mathscr{A})}$, is obtained in § 3. On its basis, in § 4 we give a complete description of the reversible extensions of C^*-dynamical systems and the corresponding reversible extensions of dynamical systems. Finally, in § 5 we look at several examples demonstrating, in particular, the relation of our results here to several classical objects of the theory of dynamical systems.

This paper is an extended and recast presentation of the e-print [6].

§ 2. Endomorphisms of commutative C^*-algebras and dynamical systems

The objects we start from are a commutative C^*-algebra $\mathscr{A}$ with unity 1 and an endomorphism $\delta \colon \mathscr{A} \to \mathscr{A}$.

Our first observation, Theorem 2.2, is that each endomorphism δ gives rise to a continuous partial map of the maximal ideal space $M = M(\mathscr{A})$ of the algebra $\mathscr{A}$.

Remark 2.1. This theorem is a special case of the general result describing the endomorphisms of semisimple Banach algebras (see [7], § 2), and we present its proof for completeness.

By the Gelfand-Naĭmark theorem the Gelfand transform establishes an isomorphism $\mathscr{A} \cong C(M)$, therefore we shall identify $\mathscr{A}$ and $C(M)$ throughout.

Theorem 2.2. *Let $\mathscr{A}$ be a commutative C^*-algebra with unity $1 \in \mathscr{A}$ and let δ be an endomorphism of $\mathscr{A}$. Consider the subset Δ of M defined by the condition*

$$\tau \in \Delta \quad \Longleftrightarrow \quad \tau(\delta(1)) = 1, \tag{10}$$

where τ is a multiplicative functional on $\mathscr{A}$. Then
(i) *the set Δ is clopen (open and closed);*
(ii) *the endomorphism δ can be defined by the formula*

$$(\delta f)(x) = \begin{cases} f(\alpha(x)), & x \in \Delta, \\ 0, & x \notin \Delta, \end{cases} \tag{11}$$

where $f \in C(M)$ and $\alpha\colon \Delta \to M$ is a continuous map.

Proof. We shall verify that Δ is clopen. Note that $\delta(1) = \delta^2(1)$, so the function $\delta(1) \in C(M)$ is an idempotent and can take only values 0 and 1. This proves that Δ is closed and open.

In terms of the Gelfand transform $a \to \widehat{a}$ we can define an action of δ on $C(M)$ by the formula

$$(\delta\widehat{a})(\tau) = \tau(\delta(a)) = \widehat{a}(\delta^*(\tau)), \qquad \tau \in M, \tag{12}$$

where $\delta^*\colon A^* \to A^*$ is the operator adjoint to δ. Clearly,

$$\delta^*(\tau) = \tau \circ \delta \tag{13}$$

is a multiplicative functional, and $\delta^*(\tau)(1) = \tau(\delta(1))$. By the definition of Δ we obtain

$$\tau \notin \Delta \quad \Longrightarrow \quad \delta^*(\tau) \equiv 0, \tag{14}$$

$$\tau \in \Delta \quad \Longrightarrow \quad \delta^*(\tau) \in M. \tag{15}$$

Now defining the map $\alpha\colon \Delta \longrightarrow M$ to be the restriction of δ^*:

$$\alpha = \delta^*\big|_{\Delta}, \tag{16}$$

we arrive at the required result.

Proposition 2.3. *Under the assumptions of Theorem 2.2 let $\alpha\colon \Delta \to M$ be the map defined by formula (11). Then the following results hold:*
(i) *if $\ker\delta = \{0\}$, then $\alpha\colon \Delta \to M$ is surjective;*
(ii) *if $\delta(1) = 1$, then $\Delta = M$.*

Proof. Let $\ker\delta = \{0\}$. Then δ is an injection which has a right inverse $\varrho\colon \delta(\mathscr{A}) \to \mathscr{A}$. Hence

$$\varrho(\delta(a)) = a, \qquad a \in \mathscr{A}. \tag{17}$$

For each $\tau \in M$ the functional $\tau \circ \varrho$ is defined on $\delta(\mathscr{A})$, is non-trivial and multiplicative. Hence it has an extension $\tau_1 \in M$ to $\mathscr{A}$ (see [8], § 2.10.2) and therefore

$$\tau_1 \circ \delta = \tau. \tag{18}$$

Then it follows from (13) and (16) that α is surjective.

Now let $\delta(1) = 1$. This means that $\tau(\delta(1)) = 1$ for each $\tau \in M$. Thus, (10) yields $\Delta = M$. The proof is complete.

In view of Theorem 2.2, the following definition of a (partial) dynamical system looks natural.

Definition 2.4. By a (*partial*) *dynamical system* we shall mean a triple (M, Δ, α), where M is a compact topological space, Δ a clopen subset of M, and $\alpha\colon \Delta \to M$ a continuous map. Unless a misunderstanding can arise, for brevity we shall also use the notation (M, α).

Remark 2.5. 1) When we talk about a dynamical system it is more usual to imply that the map α is defined on the whole of M, which is the case (M, M, α) in our terms.

2) By Theorem 2.2 each endomorphism δ of the algebra $C(M)$ defines uniquely a partial dynamical system (M, Δ, α). It is also clear that an arbitrary partial dynamical system (M, Δ, α) uniquely defines an endomorphism δ of the algebra $C(M)$ (by formula (11)). Thus, in effect Theorem 2.2 describes a bijective correspondence between endomorphisms of $C(M)$ and partial dynamical systems.

Definition 2.6. We shall say that a partial dynamical system (M, Δ, α) is *reversible* if $\alpha(\Delta)$ is an open subset of M and the map $\alpha\colon \Delta \to \alpha(\Delta)$ is a homeomorphism.

Definition 2.7. Let (M, Δ, α) be a partial dynamical system. By a *reversible extension* of this *system* we mean a partial dynamical system $(\widetilde{M}, \widetilde{\Delta}, \widetilde{\alpha})$ with the following properties:

(i) $(\widetilde{M}, \widetilde{\Delta}, \widetilde{\alpha})$ is a reversible partial dynamical system;

(ii) there exists a continuous surjective map $\Psi\colon \widetilde{M} \to M$ which realises a semiconjugacy between $(\widetilde{M}, \widetilde{\Delta}, \widetilde{\alpha})$ and (M, Δ, α), that is, which satisfies the conditions

$$\Psi(\widetilde{\Delta}) = \Delta, \qquad \Psi(\widetilde{M} \setminus \widetilde{\Delta}) = M \setminus \Delta, \qquad \Psi(\widetilde{M} \setminus \widetilde{\alpha}(\widetilde{\Delta})) = M \setminus \alpha(\Delta), \tag{19}$$

$$\alpha(\Psi(\widetilde{x})) = \Psi(\widetilde{\alpha}(\widetilde{x})), \qquad \widetilde{x} \in \widetilde{\Delta}. \tag{20}$$

The problems under consideration in this paper concern the constructions of reversible extensions of dynamical systems.

Now we return to the C^*-algebraic interpretation of the objects in question. Throughout this section $\mathscr{A}$ is a C^*-subalgebra of the algebra $L(H)$ containing the identity operator 1 and $U \in L(H)$ is an operator such that the map

$$\delta(a) = UaU^*, \qquad a \in \mathscr{A}, \tag{21}$$

is an endomorphism of $\mathscr{A}$ (which means, in particular, that U is a partial isometry).

Note that after applying the endomorphism δ n times we obtain

$$U^nU^{*n} = \delta^n(1) = \delta^n(1^2) = (\delta^n(1))^2 = (U^nU^{*n})^2,$$

which means that U^n is a partial isometry, so that U is a power partial isometry.

The explicit form (21) of δ and the equivalence (10) allow us to rewrite Theorem 2.2 for the objects under consideration in the following form.

Theorem 2.8. *Let $\mathscr{A} \subset L(H)$ be a commutative C^*-algebra containing the identity operator* 1, *and let $\delta(a) = UaU^*$ be an endomorphism of $\mathscr{A}$. Then*

(i) *the set $\Delta = \{\tau \in M : \tau(UU^*) = 1\}$ is clopen*;

(ii) *on the maximal ideal space M the endomorphism δ can be defined by the formula*

$$(\delta f)(x) = \begin{cases} f(\alpha(x)), & x \in \Delta, \\ 0, & x \notin \Delta, \end{cases} \tag{22}$$

where $f \in C(M)$ and $\alpha\colon \Delta \to M$ is a continuous map.

Moreover, Proposition 2.3 yields the following result.

Proposition 2.9. *Under the assumptions of Theorem 2.8 let $\alpha\colon \Delta \to M$ be the map defined by formula (22). Then the following results hold*:

(i) *if U is a unitary operator, then $\Delta = M$ and $\alpha\colon M \to M$ is surjective*;

(ii) *if U is an isometry, then $\alpha\colon \Delta \to M$ is surjective*;

(iii) *if U^* is an isometry, then $\Delta = M$.*

As the following theorem demonstrates, the situation described by Theorem 2.8 will be much simpler if, alongside the endomorphism δ in (21), the map

$$\delta_*(a) = U^*aU, \qquad a \in \mathscr{A}, \tag{23}$$

is also an endomorphism of the algebra $\mathscr{A}$.

Theorem 2.10. *Let $\mathscr{A}$ be a commutative C^*-subalgebra of $L(H)$ containing the unity* 1, *and assume that the maps δ and δ_* defined by formulae (21) and (23), respectively, are endomorphisms of $\mathscr{A}$. Let M be the maximal ideal space of $\mathscr{A}$. Then*

(i) *the sets $\Delta_1 = \{\tau \in M : \tau(UU^*) = 1\}$ and $\Delta_{-1} = \{\tau \in M : \tau(U^*U) = 1\}$ are clopen*;

(ii) *in terms of the algebra $C(M)$ the endomorphism δ can be defined by the formula*

$$(\delta f)(x) = \begin{cases} f(\alpha(x)), & x \in \Delta_1, \\ 0, & x \notin \Delta_1, \end{cases} \tag{24}$$

where $f \in C(M)$ and $\alpha\colon \Delta_1 \to \Delta_{-1}$ is a homeomorphism;

(iii) *the endomorphism δ_* can be defined by the formula*

$$(\delta_* f)(x) = \begin{cases} f(\alpha^{-1}(x)), & x \in \Delta_{-1}, \\ 0, & x \notin \Delta_{-1}, \end{cases} \tag{25}$$

where $f \in C(M)$.

In particular, the dynamical system (M, Δ, α) is reversible.

Proof. By Theorem 2.8 the sets $\Delta_1, \Delta_{-1} \subseteq M$ defined by

$$\tau \in \Delta_1 \iff \tau(UU^*) = 1, \tag{26}$$
$$\tau \in \Delta_{-1} \iff \tau(U^*U) = 1, \tag{27}$$

are clopen and there exist continuous maps

$$\alpha\colon \Delta_1 \to M \quad \text{and} \quad \alpha'\colon \Delta_{-1} \to M$$

for which δ and δ_*, respectively, satisfy (22). To finish the proof it is sufficient to verify that $\alpha' = \alpha^{-1}$. This follows from the relations

$$\tau \in \Delta,\ a \in A \implies \tau(\delta(\delta_*(a)) = \tau(UU^*)\tau(a)\tau(UU^*) = \tau(a), \tag{28}$$
$$\tau \in \Delta_{-1},\ a \in A \implies \tau(\delta_*(\delta(a)) = \tau(U^*U)\tau(a)\tau(U^*U) = \tau(a), \tag{29}$$

which are equivalent to the equality $(\alpha' \circ \alpha)(\tau) = (\alpha \circ \alpha')(\tau) = \tau$. The proof is complete.

In view of the above results, the following definition is natural.

Definition 2.11. By a *C^*-dynamical system* we mean a pair $(\mathscr{A}, U)$, where $\mathscr{A}$ is a commutative C^*-subalgebra of $L(H)$ containing the identity operator 1 and $U \in L(H)$ is an operator satisfying two conditions:

1) the map
$$\delta(a) := UaU^*, \qquad a \in \mathscr{A}, \tag{30}$$
is an endomorphism of the algebra $\mathscr{A}$;

2)
$$U^*U \in \mathscr{A}. \tag{31}$$

Remark 2.12. As we have already pointed out, condition 1) means that U is a partial isometry.

2.1. C^*-dynamical systems and partial dynamical systems. Now we discuss the relationship between C^*-dynamical systems and partial dynamical systems.

Let $(\mathscr{A}, U)$ be a C^*-dynamical system and $M = M(\mathscr{A})$ be the maximal ideal space of the algebra $\mathscr{A}$. The system $(\mathscr{A}, U)$ determines uniquely the partial dynamical system (M, Δ, α) described in Theorem 2.8, and condition (31) means that $\alpha(\Delta)$ is a clopen set, because the projection $U^*U \in \mathscr{A}$ can be defined in terms of the characteristic function of the set $\alpha(\Delta)$ (cf. formula (27)).

The converse result also holds. To state it we require the following definition.

Definition 2.13. Let (M, Δ, α) be a partial dynamical system. We shall say that a C^*-dynamical system $(\mathscr{A}, U)$ *corresponds* to (M, Δ, α) (or that $(\mathscr{A}, U)$ is a *covariant representation of the system* (M, Δ, α)) if the maximal ideal space $M(\mathscr{A})$ of the algebra $\mathscr{A}$ equals M and the endomorphism $\delta(\,\cdot\,) = U(\,\cdot\,)U^*$ is defined by formula (22).

Below (in § 4.2) we show that *for each partial dynamical system* (M, Δ, α) *such that* $\alpha(\Delta)$ *is a clopen set there exists a C^*-dynamical system* $(\mathscr{A}, U)$ *corresponding to* (M, Δ, α).

Remark 2.14. If there exists a C^*-dynamical system $(\mathscr{A}, U)$ corresponding to a fixed partial dynamical system (M, Δ, α), then it cannot be unique; in particular, if for a fixed algebra $\mathscr{A}$ the operator U generates an endomorphism δ of the form (22), then for each λ, $|\lambda| = 1$, the operator λU generates the same endomorphism. However we should note at this point that the objects under consideration in this paper can be described equivalently by any C^*-dynamical system corresponding to a fixed partial dynamical system (M, Δ, α) (cf. Remark 3.6).

Definition 2.15. We say that a C^*-dynamical system $(\mathscr{A}, U)$ is *reversible* if, besides condition 1) in Definition 2.11, it also satisfies the following condition: the map

$$\delta_*(a) = U^* a U, \qquad a \in \mathscr{A}, \tag{32}$$

also is an endomorphism of the algebra $\mathscr{A}$. (Clearly, condition 2) in Definition 2.11 holds automatically in this case.)

Remark 2.16. In view of Theorem 2.10, it is natural to say here that the endomorphisms δ and δ_* are mutually inverse.

Definition 2.17. Let $(\mathscr{A}, U)$ be a C^*-dynamical system. We shall say that a C^*-dynamical system $(\mathscr{B}, U)$ (with the same operator U) is a *reversible extension* of $(\mathscr{A}, U)$ if the following two conditions are fulfilled:

(i) $\mathscr{A} \subset \mathscr{B}$;

(ii) the C^*-dynamical system $(\mathscr{B}, U)$ is reversible.

In this case it is natural to call $(\mathscr{A}, U)$ a *subsystem* of the system $(\mathscr{B}, U)$.

2.2. Reversible extensions of C^*-dynamical systems and of dynamical systems. Now we discuss connections between reversible extensions of C^*-dynamical systems and reversible extensions of dynamical systems.

Let $(\mathscr{A}, U)$ be a C^*-dynamical system and $(\mathscr{B}, U)$ its reversible extension. Let (M, Δ, α) be the dynamical system determined by the system $(\mathscr{A}, U)$ in accordance with § 2.1, and let $(\widetilde{M}, \widetilde{\Delta}, \widetilde{\alpha})$ be the dynamical system determined by $(\mathscr{B}, U)$. Theorem 2.10 states that the dynamical system $(\widetilde{M}, \widetilde{\Delta}, \widetilde{\alpha})$ is reversible. Consider the map $\Psi\colon \widetilde{M} \to M$ defined by the formula

$$\Psi(\widetilde{x}) = \widetilde{x}\big|_{\mathscr{A}}; \tag{33}$$

here $\widetilde{x}\big|_{\mathscr{A}}$ is the restriction of the multiplicative functional $\widetilde{x} \in \widetilde{M}$ to the subalgebra $\mathscr{A}$. Since $1 \in \mathscr{A}$, it follows that $\Psi(\widetilde{x})$ is a non-trivial multiplicative functional, that is, $\Psi(\widetilde{x}) \in M$. Moreover, each multiplicative functional $y \in M$ extends to a multiplicative functional $\widetilde{y} \in \widetilde{M}$ (see [8], Proposition 2.10.2), therefore Ψ is surjective. Furthermore, it follows from the definitions of Ψ, Δ and $\widetilde{\Delta}$ and from Theorem 2.8,(i) that

$$\Psi(\widetilde{\Delta}) = \Delta, \qquad \Psi(\widetilde{M} \setminus \widetilde{\Delta}) = M \setminus \Delta,$$

and it follows from formulae (13) and (16) that all the other equalities in (19) and (20) also hold. Thus, Ψ defines a semiconjugacy between $(\widetilde{M}, \widetilde{\Delta}, \widetilde{\alpha})$ and (M, Δ, α), so that $(\widetilde{M}, \widetilde{\Delta}, \widetilde{\alpha})$ is an invertible extension of (M, Δ, α).

Thus, we have shown that a reversible extension of a C^*-dynamical system $(\mathscr{A}, U)$ generates a reversible extension of the partial dynamical system (M, Δ, α) determined by the former.

If we now recall Proposition 1.7 in combination with the above considerations, we see that it indicates a unique possible C^*-algebraic candidate for the role of reversible extension of the C^*-dynamical system $(\mathscr{A}, U)$, namely the C^*-dynamical system $(\overline{E_*(\mathscr{A})}, U)$.

So, in the next section we shall start our investigations by describing the maximal ideal spaces of the algebra $\overline{E_*(\mathscr{A})}$.

Remark 2.18. In connection with the questions and the objects we are looking at in this paper, we also mention some lines of research in analysis with their origins in physics that have a similar mathematical 'philosophy' in some respects.

One method of establishing relations between quantum mechanics and classical mechanics is to treat quantum mechanics as a certain non-commutative 'lift' of (commutative) classical mechanics. Then a classical dynamical system can be obtained from the corresponding quantum system by a certain average (projection) of it: classical mechanics is a 'shadow' of quantum mechanics. The reverse procedure (finding a quantum 'lift' of a classical dynamical system) is called the *quantization* of the dynamical system.

In this paper we discuss problems related to constructing reversible extensions of dynamical systems. In the light of the 'philosophy' presented above such constructions can be regarded as reversible 'quantizations' of the original irreversible dynamical systems.

Another class of problems, also with 'quantum' origins, is related to the so-called problem of 'hidden parameters'. It is connected with the conjecture that the probabilistic (and therefore, in fact, irreversible) nature of the axioms of quantum mechanics is a consequence of our ignorance (inability to find values) of some hidden parameters, and once these are found, the probabilistic quantum picture will recover its 'original' deterministic form (that is, it will transform into a picture of the same kind as classical mechanics). The problem of hidden parameters has not been solved yet and, anyway, it is clear that the number of these parameters, if they were actually discovered, would be very large, so that the 'original' deterministic form must be extremely complicated. From this point of view, the construction of reversible extensions of irreversible dynamical systems can be regarded as finding the hidden parameters, and recovering the 'original' reversible form of a dynamical system. We find these hidden parameters in this paper (they are described by formula (60) in Theorem 3.5).

§ 3. Maximal ideal space of a commutative coefficient algebra

In this section we fix a commutative C^*-subalgebra $\mathscr{A} \subset L(H)$, $1 \in \mathscr{A}$, and a partial isometry $U \in L(H)$ such that the map (21) is an endomorphism of $\mathscr{A}$ and $U^*U \in \mathscr{A}'$. Our aim is to describe the maximal ideal space $M(\overline{E_*(\mathscr{A})})$ of the coefficient C^*-algebra $\overline{E_*(\mathscr{A})} = \overline{\left\{\bigcup_{n=0}^{\infty} \delta_*^n(\mathscr{A})\right\}}$ (see Proposition 1.7) in terms of the maximal ideal space $M = M(\mathscr{A})$ of the algebra $\mathscr{A}$ and the action δ defined by formula (22).

We start by introducing the requisite objects and notation.

Let $\widetilde{x} \in M(\overline{E_*(\mathscr{A})})$ be a multiplicative linear functional on $\overline{E_*(\mathscr{A})}$. Consider the sequence of functionals $\xi_{\widetilde{x}}^n\colon \mathscr{A} \to \mathbb{C}$, $n = 0, 1, \dots$, defined by

$$\xi_{\widetilde{x}}^n(a) = \delta_*^n(a)(\widetilde{x}), \qquad a \in \mathscr{A}. \tag{34}$$

Since $\overline{E_*(\mathscr{A})} = \overline{\left\{\bigcup_{n=0}^{\infty} \delta_*^n(\mathscr{A})\right\}}$, the sequence $\xi_{\widetilde{x}}^n$ determines $\widetilde{x}$ uniquely. On the other hand, since δ_* is an endomorphism of $\overline{E_*(\mathscr{A})}$, the functionals $\xi_{\widetilde{x}}^n$ are linear and multiplicative on $\mathscr{A}$ (but it is possible that $\xi_{\widetilde{x}}^n = 0$). Thus, we either have

$$\xi_{\widetilde{x}}^n = x_n \in M, \tag{35}$$

or

$$\xi_{\widetilde{x}}^n = 0. \tag{36}$$

Obviously, the map

$$\widetilde{x} \to (\xi_{\widetilde{x}}^0, \xi_{\widetilde{x}}^1, \dots) \tag{37}$$

is injective.

Let (M, Δ, α) be a dynamical system defined by an endomorphism

$$\delta(\,\cdot\,) := U(\,\cdot\,)U^*$$

in accordance with Theorem 2.8. We shall use the following sets. Let

$$\Delta_n = \alpha^{-n}(M), \qquad n = 0, 1, 2, \dots, \tag{38}$$

so that Δ_n is the domain of definition of α^n, and let

$$\Delta_{-n} = \alpha^n(\Delta_n), \qquad n = 1, 2, \dots, \tag{39}$$

be the image of α^n.

Then we have

$$\alpha^n\colon \Delta_n \to \Delta_{-n}, \tag{40}$$

$$\alpha^n(\alpha^m(x)) = \alpha^{n+m}(x), \qquad x \in \Delta_{n+m}. \tag{41}$$

In terms of multiplicative functionals the sets Δ_n can be defined as follows: for $n > 0$,

$$\tau \in \Delta_n* \quad \iff \quad \forall_{0<k\leqslant n}\, \tau(U^k U^{*k}) = 1, \tag{42}$$

$$\tau \in \Delta_{-n}* \quad \iff \quad \exists_{\tau_n \in \Delta_n}\, \tau_n \circ \delta^n = \tau. \tag{43}$$

Note that the projections $U^k U^{*k}$ form a decreasing sequence, therefore if $\tau(U^n U^{*n}) = 1$, then $\tau(U^k U^{*k}) = 1$ for $k < n$. Hence we can write condition (42) also as

$$\tau \in \Delta_n \quad \iff \quad \tau(U^n U^{*n}) = 1. \tag{44}$$

Remark 3.1. In the case considered in Theorem 2.10 the sets Δ_{-n} are in effect the domains of definition of the map α^{-n}. Moreover, in terms of maximal ideals we have the equivalences

$$\tau \in \Delta_n \quad \iff \quad \tau(U^n U^{*n}) = 1, \tag{45}$$

$$\tau \in \Delta_{-n} \quad \iff \quad \tau(U^{*n} U^n) = 1, \tag{46}$$

where $n \geqslant 0$.

The next result is a first step of the description of the maximal ideal space $M(\overline{E_*(\mathscr{A})})$.

Theorem 3.2 (an 'upper bound' for a maximal ideal space). *Let $\mathscr{A} \subset L(H)$ be a commutative C^*-subalgebra, $1 \in \mathscr{A}$. Let $\delta(a) = UaU^*$ be an endomorphism of $\mathscr{A}$, $U^*U \in \mathscr{A}'$, let $\alpha\colon \Delta \to M(\mathscr{A})$ be the partial map defined by formula (22), and let Δ_n be the sets defined in (38) and (39). Then the maximal ideal space $M(\overline{E_*(\mathscr{A})})$ of the algebra $\overline{E_*(\mathscr{A})}$ is homeomorphic to a subset of a countable sum of disjoint sets; more precisely, the map (37) (taking account of Remarks (35) and (36)) defines a topological embedding*

$$M(\overline{E_*(\mathscr{A})}) \hookrightarrow \bigcup_{N=0}^{\infty} M_N \cup M_\infty, \tag{47}$$

where the M_N are sets of the following form:

$$M_N = \bigl\{\widetilde{x} = (x_0, x_1, \dots, x_N, 0, \dots) : x_n \in \Delta_n,\, \alpha(x_n) = x_{n-1},\, n = 1, \dots, N\bigr\}$$

and M_∞ is defined by the condition

$$M_\infty = \bigl\{\widetilde{x} = (x_0, x_1, \dots) : x_n \in \Delta_n,\, \alpha(x_n) = x_{n-1},\, n \in \mathbb{N}\bigr\}.$$

The topology in M_N, $N \in \mathbb{N} \cup \{0\}$, and M_∞ is generated by the neighbourhoods of points $\widetilde{x} \in M_N$ of the form

$$O(a_1, \dots, a_k, \varepsilon) = \bigl\{\widetilde{y} \in M_N : |a_i(x_N) - a_i(y_N)| < \varepsilon,\, i = 1, \dots, k\bigr\} \tag{48}$$

and neighbourhoods of points $\widetilde{x} \in M_\infty$ of the form

$$O(a_1, \dots, a_k, n, \varepsilon) = \biggl\{\widetilde{y} \in \bigcup_{N=n}^{\infty} M_N \cup M_\infty : |a_i(x_n) - a_i(y_n)| < \varepsilon,\, i = 1, \dots, k\biggr\}, \tag{49}$$

where $\varepsilon > 0$, $a_i \in A$ and $k, n \in \mathbb{N} \cup \{0\}$.

Proof. We pointed out above that the map (37) is injective. One of the two following situations is possible on the right-hand side of (37).

1) Assume first that some functionals $\xi^n_{\widetilde{x}}$ are trivial. Let N be the first index such that

$$\xi^{N+1}_{\widetilde{x}} \equiv 0.$$

Note that for each $n \in \mathbb{N}$ we have

$$\xi^n_{\widetilde{x}} \neq 0 \quad\Longleftrightarrow\quad \widetilde{x}(U^{*n}U^n) = 1, \tag{50}$$

and $\{U^{*n}U^n\}_{n\in\mathbb{N}}$ is a decreasing sequence of commuting projections (see [1], Proposition 3.6), that is, for $i \leqslant j$,

$$U^{*i}U^iU^{*j}U^j = U^{*j}U^jU^{*i}U^i = U^{*j}U^j.$$

Hence for each $n > N$ we obtain

$$\widetilde{x}(U^{*n}U^n) = \widetilde{x}(U^{*N+1}U^{N+1}U^{*n}U^n) = \widetilde{x}(U^{*N+1}U^{N+1})\widetilde{x}(U^{*n}U^n) = 0,$$

that is, $\xi^n_{\widetilde{x}} \equiv 0$ for $n > N$.

Since $\xi^n_{\widetilde{x}} \neq 0$ for $0 \leqslant n \leqslant N$, there exists $x_n \in M(A)$ such that

$$\xi^n_{\widetilde{x}}(a) = a(x_n), \qquad a \in \mathscr{A}, \tag{51}$$

so that the map (37) (recall (35) and (36)) has the form $\widetilde{x} \mapsto (x_0, x_1, \dots, x_N, 0, \dots)$. Moreover, since all the $U^{*i}U^i$ and U^jU^{*j} commute, it follows that

$$\begin{aligned} U^{*n}U^nU^{*n-1} &= U^{*n-1}(U^*U)(U^{n-1}U^{*n-1}) = (U^{*n-1}U^{n-1}U^{*n-1})U^*U = U^{*n}U, \\ U^{n-1}U^{*n}U^n &= (U^{n-1}U^{*n-1})(U^*U)U^{n-1} = U^*U(U^{n-1}U^{*n-1}U^{n-1}) = U^*U^n. \end{aligned}$$

Hence from the equalities $\widetilde{x}(U^{*n}U^n) = 1$, which hold for all $n = 1, \dots, N$, we conclude that for all $a \in \mathscr{A}$ and $0 < n \leqslant N$ we have

$$\begin{aligned} a(x_{n-1}) &= \xi^{n-1}_{\widetilde{x}}(a) = \widetilde{x}(\delta^{n-1}_*(a)) = \widetilde{x}(U^{*n}U^n)\widetilde{x}(\delta^{n-1}_*(a))\widetilde{x}(U^{*n}U^n) \\ &= \widetilde{x}(U^{*n}U^nU^{*n-1}aU^{n-1}U^{*n}U^n) = \widetilde{x}(U^{*n}UaU^*U^n) = \widetilde{x}(\delta^n_*(\delta(a))) \\ &= \xi^n_{\widetilde{x}}(\delta(a)) = \delta(a)(x_n) = a(\alpha(x_n)), \end{aligned}$$

where we have used (51) and (22) in the last two equalities. Since the algebra $\mathscr{A}$ separates the points in $M(\mathscr{A})$, it follows that

$$\alpha(x_n) = x_{n-1}, \qquad n \in \mathbb{N}. \tag{52}$$

Consequently,

$$x_n \in \Delta_n, \qquad 0 \leqslant n \leqslant N, \tag{53}$$

and therefore

$$\widetilde{x} \mapsto (x_0, x_1, \dots, x_N, 0, \dots) \in M_N.$$

2) Now assume that for each $n \in \mathbb{N}$ we have

$$\xi^n_{\widetilde{x}} \neq 0,$$

that is, $\xi^n_{\widetilde{x}} \in M(\mathscr{A})$. Selecting $x_n \in M(\mathscr{A})$ from (51) we see that the map (37) has the form $\widetilde{x} \mapsto (x_0, x_1, x_2, \dots)$.

Using the same arguments as in case 1) we establish the equality

$$\alpha(x_n) = x_{n-1}, \qquad n \geqslant 1. \tag{54}$$

From the definition of Δ_n, $n \in \mathbb{Z}$, we obtain

$$x_n \in \Delta_n, \qquad n \in \mathbb{N}. \tag{55}$$

Hence

$$\widetilde{x} \mapsto (x_0, x_1, x_2, \dots) \in M_\infty.$$

Now recall that the map (37) is injective and, as shown above, the right-hand side of (37) belongs to M_N or M_∞ (depending on $\widetilde{x}$). This means that (37) defines an embedding (47).

Finally, we note that $M(\overline{E_*(\mathscr{A})})$ carries the weak-* topology. Hence a point $\widetilde{x} = (x_0, x_1, \dots) \in M_\infty$ has a fundamental system of neighbourhoods of the form

$$O(b_1, \dots, b_k, \varepsilon) = \bigl\{\widetilde{y} \in M(\overline{E_*(\mathscr{A})}) : |b_i(\widetilde{x}) - b_i(\widetilde{y})| < \varepsilon,\ i = 1, \dots, k\bigr\}, \tag{56}$$

where $b_i \in \overline{E_*(\mathscr{A})}$, $\varepsilon > 0$. Since $\overline{E_*(\mathscr{A})} = \overline{\{\bigcup_{n=0}^{\infty} \delta_*^n(\mathscr{A})\}}$, it is sufficient to take $b_i = \delta_*^n(a_i)$, $a_i \in \mathscr{A}$, $i = 0, \dots, k$, in (56) so that we obtain

$$\begin{aligned} O(b_1, \dots, b_k, \varepsilon) &= \bigl\{\widetilde{y} \in M(\overline{E_*(\mathscr{A})}) : |\delta_*^n(a_i)(\widetilde{x}) - \delta_*^n(a_i)(\widetilde{y})| < \varepsilon,\ i = 1, \dots, k\bigr\} \\ &= \bigl\{\widetilde{y} = (y_0, y_1, \dots) \in M(\overline{E_*(\mathscr{A})}) : |a_i(x_n) - a_i(y_n)| < \varepsilon,\ i = 1, \dots, k\bigr\}. \end{aligned}$$

Setting $O(a_1, \dots, a_k, n, \varepsilon) = O(b_1, \dots, b_k, \varepsilon) \cap \bigl(\bigcup_{N \geqslant n} M_N \cup M_\infty\bigr)$ we obtain the neighbourhood basis required in the theorem.

For $\widetilde{x} = (x_0, x_1, \dots, x_N, 0, \dots) \in M_N$ we set

$$O(a_1, \dots, a_k, \varepsilon) = O(a_1, \dots, a_k, N, \varepsilon) \cap M_N.$$

Thus, formulae (48) and (49) define a neighbourhood basis of a point

$$\widetilde{x} \in M(\overline{E_*(\mathscr{A})}) \hookrightarrow \bigcup_{N=0}^{\infty} M_N \cup M_\infty,$$

which completes the proof of the theorem.

Remark 3.3. It is useful to note that the set of operators of the form

$$b = a_0 + \delta_*(a_1) + \dots + \delta_*^N(a_N),$$

where $a_0, a_1, \dots, a_N \in A$, is dense in $\overline{E_*(\mathscr{A})}$ (see [1], Proposition 3.8). Hence from (51), the definition of the functionals generating the sequence $\widetilde{x} = (x_0, \dots)$, we conclude that

$$b(\widetilde{x}) = a_0(x_0) + a_1(x_1) + \dots + a_N(x_N)$$

for $\widetilde{x} = (x_0, \dots) \in \bigcup_{n=N}^{\infty} M_n \cup M_\infty$ and

$$b(\widetilde{x}) = a_0(x_0) + a_1(x_1) + \dots + a_n(x_k),$$

for $\widetilde{x} = (x_0, \dots, x_k, 0, \dots) \in \bigcup_{n=0}^{N} M_n$.

The theorem just proved provides an 'upper bound' for the space $M(\overline{E_*(\mathscr{A})})$. Theorem 3.4, which follows, yields a 'lower bound'. Before stating it we emphasize that the previous theorem claims that each $\widetilde{x} \in M(\overline{E_*(\mathscr{A})})$ generates (defines uniquely) a point in M_N or M_∞. On the other hand, for a sequence $(x_0, x_1, \dots, x_N, 0, \dots) \in M_N$ or $(x_0, x_1, \dots, x_n, \dots) \in M_\infty$, we do not know in advance that it is generated by some $\widetilde{x} \in M(\overline{E_*(\mathscr{A})})$ (this only holds if the sequence has the form (37)). Theorem 3.4 says that all sequences in the subset $\widehat{M}_N$ of M_N (which may be much smaller than M_N in the general case) and all sequences in M_∞ are indeed generated by some elements $\widetilde{x} \in M(\overline{E_*(\mathscr{A})})$.

Theorem 3.4 (a 'lower bound' for a maximal ideal space). *Let $\mathscr{A} \subset L(H)$ be a commutative C^*-algebra, $1 \in \mathscr{A}$, and let $U \in L(H)$ be a partial isometry such that $U^*U \in \mathscr{A}'$. By Theorem 3.2, $M(\overline{E_*(\mathscr{A})})$ can be regarded as a subset of the space $\bigcup_{N \geqslant 0} M_N \cup M_\infty$; on the other hand*

$$\bigcup_{N=0}^{\infty} \widehat{M}_N \cup M_\infty \subset M(\overline{E_*(\mathscr{A})}), \tag{57}$$

where the $\widehat{M}_N$ *are the sets of the following form*:

$$\widehat{M}_N = \bigl\{\widetilde{x} = (x_0, x_1, \dots, x_N, 0, \dots) : x_N \in \Delta_N,\, x_N \notin \Delta_{-1},\, \alpha(x_n) = x_{n-1}\bigr\}$$

and $M_\infty = \bigl\{\widetilde{x} = (x_0, x_1, \dots) : x_n \in \Delta_n,\, \alpha(x_n) = x_{n-1},\, n \geqslant 1\bigr\}$.

Proof. First, we will show that $M_\infty \subset M(\overline{E_*(\mathscr{A})})$. Let $(x_0, x_1, \dots)$ be a sequence of elements of $M(\mathscr{A})$ such that $\alpha(x_n) = x_{n-1}$ for each $n \geqslant 1$. We claim that there exists a multiplicative linear functional $\widetilde{x} \in M(\overline{E_*(\mathscr{A})})$ generating this sequence (that is, the equalities (51) hold for $n = 0, 1, 2, \dots$ with $\xi^n_{\widetilde{x}}$ defined by (34)).

In fact, consider the sets

$$\widetilde{X}^n = \bigl\{\widetilde{x} \in M(\overline{E_*(\mathscr{A})}) : \forall_{a\in A}\, \xi^n_{\widetilde{x}}(a) = a(x_n)\bigr\}, \qquad n = 0, 1, \dots, \tag{58}$$

with $\xi^n_{\widetilde{x}}$ defined by (34). Then

1) $\widetilde{X}^n \neq \varnothing$. This is because (58) is the set of all extensions of multiplicative functionals from $\delta^n_*(\mathscr{A})$ to $\overline{E_*(\mathscr{A})}$. As is known, each multiplicative functional on a commutative C^*-subalgebra can be extended to a multiplicative functional on any larger commutative C^*-algebra (see [8], § 2.10.2);
2) $\widetilde{X}^n$ is closed, as follows from the definition of weak-* convergence;
3) $\widetilde{X}^0 \supset \widetilde{X}^1 \supset \dots \supset \widetilde{X}^n \supset \cdots$. Indeed, if $\widetilde{x} \in \widetilde{X}^n$, then

$$a(x_{n-1}) = a(\alpha(x_n)) = \delta(a)(x_n) = \xi^n_{\widetilde{x}}(\delta(a)) = \xi^{n-1}_{\widetilde{x}}(a), \qquad a \in \mathscr{A},$$

that is, $\widetilde{x} \in \widetilde{X}^{n-1}$.

Thus, the family of sets $\widetilde{X}^n$ forms a decreasing sequence of non-empty compact sets, and therefore $\bigcap_{n=0}^\infty \widetilde{X}^n \neq \varnothing$. It follows from the definition of the $\widetilde{X}^n$ that each point $\widetilde{x} \in \bigcap_{n=0}^\infty \widetilde{X}^n$ generates the same sequence

$$(\xi^0_{\widetilde{x}}, \xi^1_{\widetilde{x}}, \xi^2_{\widetilde{x}}, \dots) = (x_0, x_1, x_2, \dots).$$

However, as already pointed out, the map

$$\widetilde{x} \to (\xi^0_{\widetilde{x}}, \xi^1_{\widetilde{x}}, \xi^2_{\widetilde{x}}, \dots)$$

is injective, and therefore the set

$$\bigcap_{n=0}^\infty \widetilde{X}^n = \{\widetilde{x}\}$$

reduces to a singleton.

Now let $(x_0, x_1, \dots, x_N, 0, \dots) \in \widehat{M}_N$. Consider the sets $\widetilde{X}^n$ defined by (58) for $n = 0, 1, \dots, N$. The above arguments show that these are non-empty sets forming a decreasing sequence. Let $\widetilde{x} \in \widetilde{X}^N$. To identify $\widetilde{x}$ with the sequence $(\xi^0_{\widetilde{x}}, \xi^1_{\widetilde{x}}, \dots, \xi^N_{\widetilde{x}}, 0, \dots) = (x_0, x_1, \dots, x_N, 0, \dots)$ defined in (37) it is sufficient to show that

$$\xi^{N+1}_{\widetilde{x}} \equiv 0.$$

Assume by contradiction that $\xi^{N+1}_{\widetilde{x}} \neq 0$. Then we see from part 1) of the proof of Theorem 3.2 that $\widetilde{x} = (x_0, x_1, \dots, x_N, x_{N+1}, \dots)$, where $\alpha(x_{N+1}) = x_N$, which contradicts the relation $x_N \notin \Delta_{-1} = \alpha(\Delta_1)$. The proof is complete.

In the general case neither of the inclusions (47) and (57) in Theorems 3.2 and 3.4 is an equality. However, the following result demonstrates that in the case when $\mathscr{A}$ contains an element $\delta_*(1) = U^*U$ we have equality in (57), so this formula describes completely the maximal ideal space $M(\overline{E_*(\mathscr{A})})$.

Theorem 3.5 (a maximal ideal space: a complete description). *Let $\mathscr{A} \subset L(H)$ be a commutative C^*-algebra with unity. Let $\delta(\,\cdot\,) = U(\,\cdot\,)U^*$ be an endomorphism of $\mathscr{A}$ and, moreover, let*

$$U^*U \in \mathscr{A}. \tag{59}$$

Then the maximal ideal space $M(\overline{E_*(\mathscr{A})})$ of $\overline{E_*(\mathscr{A})}$ is homeomorphic to the countable disjoint sum of the clopen sets $\widehat{M}_N$ and the closed set M_∞ (some of these sets may be empty)

$$M(\overline{E_*(\mathscr{A})}) = \bigcup_{N=0}^{\infty} \widehat{M}_N \cup M_\infty, \tag{60}$$

where

$$\widehat{M}_N = \bigl\{\widetilde{x} = (x_0, x_1, \dots, x_N, 0, \dots) : x_N \in \Delta_N,\ x_N \notin \Delta_{-1},\ \alpha(x_n) = x_{n-1}\bigr\},$$
$$M_\infty = \bigl\{\widetilde{x} = (x_0, x_1, \dots) : x_n \in \Delta_n \cap \Delta_{-\infty},\ \alpha(x_n) = x_{n-1},\ n \geqslant 1\bigr\}.$$

The topology on $\bigcup_{N=0}^{\infty} \widehat{M}_N \cup M_\infty$ is defined by a fundamental system of neighbourhoods of points $\widetilde{x} \in \widehat{M}_N$ of the form

$$O(a_1, \dots, a_k, \varepsilon) = \bigl\{\widetilde{y} \in \widehat{M}_N : |a_i(x_N) - a_i(y_N)| < \varepsilon,\ i = 1, \dots, k\bigr\} \tag{61}$$

and of neighbourhoods of points $\widetilde{x} \in M_\infty$ of the form

$$O(a_1, \dots, a_k, n, \varepsilon) = \biggl\{\widetilde{y} \in \bigcup_{N=n}^{\infty} \widehat{M}_N \cup M_\infty : |a_i(x_n) - a_i(y_n)| < \varepsilon,\ i = 1, \dots, k\biggr\},$$

where $\varepsilon > 0$, $a_i \in A$ and $k, n \in \mathbb{N} \cup \{0\}$.

Proof. It follows from Theorems 3.2 and 3.4 that to prove equality (60) it is sufficient to prove the implication

$$\widetilde{x} \in M_N \quad \Longrightarrow \quad \widetilde{x} \in \widehat{M}_N$$

for each $\widetilde{x} \in M(\overline{E_*(A)})$. Assume on the contrary that $\widetilde{x} \in M_N$ and $\widetilde{x} \notin \widehat{M}_N$. Then $\widetilde{x} = (x_0, x_1, \dots, x_N, 0, \dots)$, $x_n \in M(\mathscr{A})$, and there exists $x_{N+1} \in \Delta_1 \subset M(\mathscr{A})$ such that $\alpha(x_{N+1}) = x_N$. It follows from (51) and (34), which define the functionals x_n and $\xi^n_{\widetilde{x}}$ respectively, that $\widetilde{x}(U^{*n}aU^n) = a(x_n)$ for all $a \in A$ and $n = 0, \dots, N$ and that $\widetilde{x}(U^{*n}aU^n) = 0$ for $n > N$. In particular,

$$\widetilde{x}(U^{*N}U^N) = 1 \quad \text{and} \quad \widetilde{x}(U^{*N+1}U^{N+1}) = 0.$$

By formula (22) we obtain

$$\widetilde{x}(U^{*N}aU^N) = a(x_N) = a(\alpha(x_{N+1})) = \delta(a)(x_{N+1}), \qquad a \in A.$$

Setting $a = U^*U \in A$ we see that $\delta(U^*U)(x_{N+1}) = 0$. On the other hand, for $\delta(U^*U)(x_{N+1})$ we have

$$\delta(U^*U)(x_{N+1}) = x_{N+1}(UU^*UU^*) = x_{N+1}(UU^*)x_{N+1}(UU^*) = 1.$$

We have arrived at a contradiction, which proves (60).

The sets $\widehat{M}_N$ are closed and open because of the following:

$$\widetilde{x} \in M_N \iff \widetilde{x} \in \widehat{M}_N \iff \bigl\{\widetilde{x}(U^{*N}U^N) = 1,\, \widetilde{x}(U^{*N+1}U^{N+1}) = 0\bigr\}.$$

On the other hand, since the $\widehat{M}_N$ are open, the set M_∞ is closed, which completes the proof of the theorem.

Remark 3.6. Theorem 3.5 demonstrates that the maximal ideal space $M(\overline{E_*(\mathscr{A})})$ in reality only depends on the pair $(\mathscr{A},\delta)$ and is independent of the choice of the operator U defining the endomorphism δ (provided that (59) holds). In other words, if (M,Δ,α) is a partial dynamical system and $\alpha(\Delta)$ is a closed set, then the space $M(\overline{E_*(\mathscr{A})})$ depends only on the system (M,Δ,α) itself, but not on the choice of its covariant representation $(\mathscr{A},U)$ (see § 2.1 and Definition 2.13).

Remark 3.7. In fact, the above theorem gives us a key to obtaining a complete description of the space $M(\overline{E_*(\mathscr{A})})$ in the general situation when (59) does not hold. Indeed, if $U^*U \notin \mathscr{A}$, then we can consider the C^*-algebra $\mathscr{A}_1 = \langle \mathscr{A}, U^*U\rangle$ generated by $\mathscr{A}$ and U^*U. Since

$$\delta(U^*U) = UU^*UU^* = \delta(1)\delta(1) = \delta(1) \in \mathscr{A},$$

it follows that $\delta\colon \mathscr{A}_1 \to \mathscr{A}$. Applying the above theorem to the algebra $\mathscr{A}_1$ and the operator U we obtain a complete description of the space $M(\overline{E_*(\mathscr{A}_1)})$. However,

$$\overline{E_*(\mathscr{A}_1)} = \overline{E_*(\mathscr{A})},$$

and therefore

$$M(\overline{E_*(\mathscr{A}_1)}) = M(\overline{E_*(\mathscr{A})}).$$

The results obtained can be refined (simplified) if we know the partial isometry U has some more special properties. For instance, if U is an isometry: $U^*U = 1$, then Proposition 2.9 shows that the map α generated by the endomorphism δ is surjective and therefore all the sets $\widehat{M}_N$ are empty. We arrive at the following consequence of Theorem 3.5.

Corollary 3.8. *Let $A \subset L(H)$ be a commutative C^*-algebra with unity and let U be an isometry such that $\delta(\,\cdot\,) = U(\,\cdot\,)U^*$ is an endomorphism of $\mathscr{A}$. Then the map $\alpha\colon \Delta \to M(\mathscr{A})$ defined by formula (22) is surjective and the maximal ideal space $M(\overline{E_*(\mathscr{A})})$ of the algebra $\overline{E_*(A)}$ has the following form:*

$$M(\overline{E_*(\mathscr{A})}) = \bigl\{\widetilde{x} = (x_0, x_1, \dots) : x_n \in \Delta_n,\, \alpha(x_{n+1}) = x_n,\, n \geqslant 0\bigr\}. \qquad (62)$$

Each point $\widetilde{x}$ has a fundamental system of neighbourhoods

$$O(a_1,\dots,a_k,n,\varepsilon) = \bigl\{\widetilde{y} \in M(\overline{E_*(A)}) : |a_i(x_n) - a_i(y_n)| < \varepsilon,\, i = 1,\dots,k\bigr\},$$

where $n \in \mathbb{N}\cup\{0\}$, $\varepsilon > 0$ and $a_i \in \mathscr{A}$, $1 \leqslant i \leqslant k$.

Remark 3.9. Note that if δ is an automorphism (that is, when $\Delta = M(\mathscr{A})$ and α is a homeomorphism), then the map $\Psi\colon M(\overline{E_*(\mathscr{A})}) \to M(\mathscr{A})$ of the form $\Psi(\widetilde{x}) = x_0$ defines a homeomorphism between $M(\overline{E_*(\mathscr{A})})$ and $M(\mathscr{A})$.

Remark 3.10. We also underline the fundamental connection between the objects described in Theorem 3.5 and the inverse (projective) limits of sequences of spaces with maps: the term M_∞ on the right-hand side of (60) is just the inverse (projective) limit of the sequence

$$M \xleftarrow{\alpha} \Delta_1 \xleftarrow{\alpha} \Delta_2 \xleftarrow{\alpha} \cdots . \tag{63}$$

In particular, if α is surjective, that is, if we are in the situation described by Corollary 3.8, then

$$M(\overline{E_*(\mathscr{A})}) = M_\infty = \varprojlim(M,\alpha),$$

where on the right-hand side we have the inverse limit of the sequence (63).

§ 4. Extensions of dynamical systems and of algebra endomorphisms

4.1. An extension of a C^*-dynamical system. Now that we have the description of the maximal ideal space of the algebra $\overline{E_*(\mathscr{A})}$ given in the previous section, we can easily obtain a description of a reversible extension of the dynamical system (M,Δ,α) corresponding to the endomorphism δ (see Theorem 2.8). We give this description below, in Theorem 4.1. To state it we require the following objects, which we define using the notation of Theorem 3.5:

$$\widetilde{M} := M(\overline{E_*(\mathscr{A})}) = \bigcup_{N=0}^{\infty} \widehat{M}_N \cup M_\infty, \tag{64}$$

$$\widetilde{\Delta} := \{\widetilde{x} \in \widetilde{M} : x_0 \in \Delta\}, \tag{65}$$

$$\widetilde{\alpha}\colon \widetilde{\Delta} \to \widetilde{M}, \qquad \widetilde{\alpha}(x_0,x_1,\dots) := (\alpha(x_0),x_0,x_1,\dots), \tag{66}$$

where $(x_0,x_1,\dots) = \widetilde{x}$.

We point out that it follows from (65) and (66) that

$$\widetilde{\alpha}(\widetilde{\Delta}) = \{\widetilde{x} \in \widetilde{M} : x_0 \in \alpha(\Delta)\}. \tag{67}$$

Furthermore, we shall consider the surjective map $\Psi\colon \widetilde{M} \to M$ defined by the formula

$$\Psi(\widetilde{x}) := x_0. \tag{68}$$

We see from the explicit form of Δ and α that the map Ψ satisfies conditions (19) and (20), that is, Ψ brings about a semiconjugacy between $(\widetilde{M},\widetilde{\Delta},\widetilde{\alpha})$ and (M,Δ,α).

Theorem 4.1. *Let $\mathscr{A}$, U and δ be the same algebra, operator, and endomorphism, respectively, as in Theorem 3.5, and assume that in terms of the maximal ideal space M of $\mathscr{A}$ the endomorphism δ can be defined by formula (22). Then the map δ: $\overline{E_*(\mathscr{A})} \to \overline{E_*(\mathscr{A})}$, $\delta(\,\cdot\,) = U(\,\cdot\,)U^*$, is an endomorphism of the algebra $\overline{E_*(\mathscr{A})}$ (an extension of δ) and $\delta_*\colon \overline{E_*(\mathscr{A})} \to \overline{E_*(\mathscr{A})}$, $\delta_*(\,\cdot\,) = U^*(\,\cdot\,)U$, is also an endomorphism of the algebra $\overline{E_*(\mathscr{A})}$ (inverse to δ).*

In terms of the space $\widetilde{M}$ in (64) the endomorphism δ is defined by the formula

$$(\delta f)(\widetilde{x}) = \begin{cases} f(\widetilde{\alpha}(\widetilde{x})), & \widetilde{x} \in \widetilde{\Delta}, \\ 0, & \widetilde{x} \notin \widetilde{\Delta}, \end{cases} \tag{69}$$

where $f \in C(\widetilde{M})$, $\widetilde{\Delta}$ *and* $\widetilde{\alpha}$ *are defined by formulae* (65) *and* (66); *furthermore,* $\widetilde{\alpha}\colon \widetilde{\Delta} \to \widetilde{\alpha}(\widetilde{\Delta})$ *is a homeomorphism,* $\widetilde{\Delta}$ *and* $\widetilde{\alpha}(\widetilde{\Delta})$ *are clopen subsets of* $\widetilde{M}$; *and the endomorphism* δ_* *can be defined by the formula*

$$(\delta_* f)(\widetilde{x}) = \begin{cases} f(\widetilde{\alpha}^{-1}(\widetilde{x})), & \widetilde{x} \in \widetilde{\alpha}(\widetilde{\Delta}), \\ 0, & \widetilde{x} \notin \widetilde{\alpha}(\widetilde{\Delta}), \end{cases} \tag{70}$$

where $f \in C(\widetilde{M})$, $\widetilde{\alpha}(\widetilde{\Delta}) = \{\widetilde{x} \in \widetilde{M} : x_0 \in \alpha(\Delta)\}$ *and for* $\widetilde{x} = (x_0, x_1, \dots) \in \widetilde{\alpha}(\widetilde{\Delta})$ *the map* $\widetilde{\alpha}^{-1}$ *has the form*

$$\widetilde{\alpha}^{-1}(x_0, x_1, \dots) = (x_1, \dots). \tag{71}$$

In particular, the dynamical system $(\widetilde{M}, \widetilde{\Delta}, \widetilde{\alpha})$ *is a reversible extension of the dynamical system* (M, Δ, α) *in the sense of Definition* 2.7, *where* Ψ *is equal to the surjective map* (68), *and the* C^**-dynamical system* $(\overline{E_*(\mathscr{A})}, U)$ *is the reversible extension of the* C^**-dynamical system* $(\mathscr{A}, U)$ *that corresponds to the dynamical system* $(\widetilde{M}, \widetilde{\Delta}, \widetilde{\alpha})$ *in the sense of Definition* 2.13.

Proof. By Proposition 1.7 the maps δ and δ_* are endomorphisms of the algebra $\overline{E_*(\mathscr{A})}$. Therefore, in accordance with Theorem 2.10, to complete the proof it remains to show that $\widetilde{\Delta}$ and $\widetilde{\alpha}$ in formula (69) are indeed defined by formulae (65) and (66). Then (70) and (71) follow automatically.

By Theorem 2.10 the set $\widetilde{\Delta}$ in formula (69) is defined by the condition

$$\widetilde{\Delta} = \{\widetilde{x} \in \widetilde{M} : \widetilde{x}(UU^*) = 1\}. \tag{72}$$

It follows from the explicit expression for $\widetilde{x}$ (see (34)–(37)) and the description of the set Δ given in Theorem 2.8 that

$$\widetilde{x}(UU^*) = 1 \quad\Longleftrightarrow\quad x_0(UU^*) = 1 \quad\Longleftrightarrow\quad x_0 \in \Delta.$$

Thus, $\widetilde{\Delta}$ defined by condition (72) coincides with the set defined by (65).

It only remains to demonstrate that the map $\widetilde{\alpha}$ in (69) has the form (66).

Note that for $a \in \mathscr{A}$ and $n \geqslant 1$ we have

$$\delta(\delta_*^n(a)) = UU^{*n}aU^nU^* = \delta(1)U^{*n-1}aU^{n-1}\delta(1) = \delta(1)\delta_*^{n-1}(a). \tag{73}$$

We fix an arbitrary functional

$$\widetilde{x} \in \widetilde{\Delta}, \qquad \widetilde{x} =: (x_0, x_1, \dots), \quad x_0 \in \Delta,$$

and set

$$(y_0, y_1, \dots) := \widetilde{\alpha}(\widetilde{x}). \tag{74}$$

Next we calculate the y_n, $n = 0, 1, \dots$. From the explicit expressions for the y_n (formulae (34)–(37) for $\widetilde{\alpha}(\widetilde{x})$), formulae (69), (73) and the explicit expressions for the x_n, $n = 0, 1, \dots$, we obtain the following equalities for $a \in \mathscr{A}$ and $n \geqslant 1$:

$$\begin{aligned} y_n(a) &= \delta_*^n(a)(\widetilde{\alpha}(\widetilde{x})) = \delta(\delta_*^n(a))(\widetilde{x}) = \bigl(\delta(1)\delta_*^{n-1}(a)\bigr)(\widetilde{x}) \\ &= \delta(1)(\widetilde{x})\delta_*^{n-1}(a)(\widetilde{x}) = \delta_*^{n-1}(a)(\widetilde{x}) = a(x_{n-1}). \end{aligned}$$

Hence $y_n = x_{n-1}$, $n = 1, 2, \dots$.

Using the explicit expression for y_0 and, again, formula (69), for each $a \in \mathscr{A}$ we obtain

$$a(y_0) = a(\widetilde{\alpha}(\widetilde{x})) = \delta(a)(\widetilde{x}) = \delta(a)(x_0) = a(\alpha(x_0)),$$

where we have used (22) in the last equality. Hence $y_0 = \alpha(x_0)$.

Thus, we have explicitly found the map $\widetilde{\alpha}$ in formula (69), which completes the proof of the theorem.

So far, we have assumed (in Theorems 3.5 and 4.1) that there exists an operator U defining the endomorphism δ, that is, we have assumed the existence of a C^*-dynamical system $(\mathscr{A}, U)$ corresponding to the dynamical system (M, Δ, α). On the other hand, as Theorem 3.5 shows, if $U^*U \in \mathscr{A}$ (see (59)), then the maximal ideal space of the algebra $\overline{E_*(\mathscr{A})}$ extending the algebra $\mathscr{A}$ does not in fact depend on the choice of the operator U (provided that it exists). In terms of the dynamical system (M, Δ, α) the condition $U^*U \in \mathscr{A}$ means simply that the set $\alpha(\Delta)$ is open (and therefore clopen). The following question is natural: does there always exist a C^*-dynamical system $(\mathscr{A}, U)$ corresponding to a fixed dynamical system (M, Δ, α)?

The simple construction below demonstrates that if the set $\alpha(\Delta)$ is open, then there exists a C^*-dynamical system $(\mathscr{B}, U)$ corresponding to the reversible extension $(\widetilde{M}, \widetilde{\Delta}, \widetilde{\alpha})$ of (M, Δ, α) described in Theorem 4.1 and with some subsystem $(\mathscr{A}, U)$ corresponding to the dynamical system (M, Δ, α).

4.2. A representation of an extension of a C^*-dynamical system corresponding to an extension of a dynamical system. Let (M, Δ, α) be a partial dynamical system such that $\alpha(\Delta)$ is an open set. Let $(\widetilde{M}, \widetilde{\Delta}, \widetilde{\alpha})$ be its reversible extension, where $\widetilde{M}$, $\widetilde{\Delta}$ and $\widetilde{\alpha}$ are defined by formulae (64), (65) and (66), respectively, and the semiconjugacy Ψ is defined by formula (68). Consider the Hilbert space $l^2(\widetilde{M})$ (we take a discrete measure on $\widetilde{M}$, with each point having measure one). In $l^2(\widetilde{M})$ we define a faithful representation of the algebra $\mathscr{B} := C(\widetilde{M})$ by the operators of multiplication by functions $a \in C(\widetilde{M})$. Let $U\colon l^2(\widetilde{M}) \to l^2(\widetilde{M})$ be the partial isometry defined by the formula

$$(Uf)(\widetilde{x}) = \begin{cases} f(\widetilde{\alpha}(\widetilde{x})), & \widetilde{x} \in \widetilde{\Delta}, \\ 0, & \widetilde{x} \notin \widetilde{\Delta}. \end{cases} \tag{75}$$

This formula implies that

$$UaU^* = \delta(a), \quad U^*aU = \delta_*(a), \qquad a \in \mathscr{B},$$

where δ and δ_* are defined by (69) and (70), respectively.

Consider the subalgebra $\mathscr{A} \subset B$ of functions depending only on the first coordinate x_0 of a point $\widetilde{x} \in \widetilde{M}$, that is, $\mathscr{A} = \{a \in C(\widetilde{M}) : a(\widetilde{x}) = a(x_0)\}$, where $(x_0, x_1, \dots) = \widetilde{x} \in \widetilde{M}$. Clearly, $M(\mathscr{A}) = M$. It follows from (65), (66) and the definition of the operator U that

$$U(\,\cdot\,)U^*\colon \mathscr{A} \to \mathscr{A}, \qquad UaU^* = \delta(a), \quad a \in \mathscr{A},$$

where $\delta\colon C(M) \to C(M)$ takes the form in (11). Moreover, the operator $U^*U \in \mathscr{A}$ is the projection corresponding to the characteristic function of the set $\alpha(\Delta)$.

Thus, the C^*-dynamical system $(\mathscr{B}, U)$ is an extension of the C^*-dynamical system $(\mathscr{A}, U)$ corresponding to the reversible extension $(\widetilde{M}, \widetilde{\Delta}, \widetilde{\alpha})$ of the dynamical system (M, Δ, α).

Remark 4.2. 1) It follows from our analysis, in particular, that the condition '$\alpha(\Delta)$ is an open set' is necessary and sufficient for a C^*-dynamical system $(\mathscr{A}, U)$ corresponding to the dynamical system (M, Δ, α) to exist; moreover, it has the property $U^*U \in \mathscr{A}$.

2) For more details of the interior structure and more properties of extensions of dynamical and C^*-dynamical systems and, in particular, for an analysis of the properties of the operators $U^{*n}U^n$ the reader can consult [9].

3) A construction of the crossed product generated by endomorphisms of commutative C^*-algebras was put forward in [4], based on the method for constructing reversible extensions of dynamical system developed here.

§ 5. Examples

In this section we give several examples revealing some relationships between the algebras considered in this paper and several classical objects from the theory of dynamical systems. In reality, these examples only involve algebras described in terms of M_∞, that is, they are far from exhausting even the typical C^*-algebras which arise naturally in the course of reversible extension (whose maximal ideal space, in accordance with Theorem 3.5, can also contain countably many summands of other types).

As noted already (see Remark 3.10), in effect the set M_∞ is the inverse limit for the corresponding sequence of spaces with maps. Many authors have investigated how to describe such spaces; many results in this area can be found in [9]–[18], for instance; furthermore, a description of reversible extensions of unimodal maps of a closed line segment was obtained in [19] using the techniques developed in this paper.

More than anything, the importance of the examples below is that we actually obtain some descriptions of the maximal ideal spaces and of C^*-dynamical systems that are reversible extensions of the original non-reversible systems.

Dynamical systems in this section are defined on the whole of the space, therefore we denote them by pairs of the form (M, α), rather than by triples (M, M, α).

Example 5.1 (a topological Markov chain). We recall the construction of a topological Markov chain. Let $A = (A(i,j))_{i,j\in\{1,\dots,N\}}$ be a square matrix with entries from the set $\{0,1\}$ and assume that A contains no zero rows. We associate with A two dynamical systems, (X_A, σ_A) and $(\overline{X}_A, \overline{\sigma}_A)$.

The *one-sided Markov shift* σ_A acts on the compact space

$$X_A = \bigl\{x = (\xi_k)_{k\in\mathbb{N}} \in \{1,\dots,N\}^{\mathbb{N}} : A(\xi_k, \xi_{k+1}) = 1,\ k \in \mathbb{N}\bigr\}$$

(the topology on X_A is inherited from the Cantor space $\{1,\dots,N\}^{\mathbb{N}}$) by the rule

$$(\sigma_A x)_k = \xi_{k+1}, \qquad k \in \mathbb{N}, \quad x \in X_A. \tag{76}$$

The map σ_A is surjective if and only if A contains no zero columns.

The *two-sided Markov shift* $\overline{\sigma}_A$ acts on the compact space

$$\overline{X}_A = \big\{(\xi_k)_{k\in\mathbb{Z}} \in \{1,\dots,N\}^{\mathbb{Z}} : A(\xi_k,\xi_{k+1}) = 1,\, k\in\mathbb{Z}\big\}$$

by the rule

$$(\overline{\sigma}_A x)_k = \xi_{k+1}, \qquad k\in\mathbb{Z}, \quad x\in\overline{X}_A. \tag{77}$$

The dynamical systems (X_A,σ_A) and $(\overline{X}_A,\overline{\sigma}_A)$ are also called *topological Markov chains.*

Proposition 5.2. *Assume that the matrix A contains no zero columns and let $(\widetilde{X}_A,\widetilde{\sigma}_A)$ be the reversible extension of a dynamical system (X_A,σ_A) that is described in Theorem 4.1. Then*

$$(\widetilde{X}_A,\widetilde{\sigma}_A) \cong (\overline{X}_A,\overline{\sigma}_A).$$

Proof. Since A contains no zero columns, the map σ_A is surjective, therefore in terms of Theorem 3.5 we have $\widetilde{X} = X_\infty$ (and furthermore, $U^*U = 1$). Thus,

$$\widetilde{X} = \big\{\widetilde{x} = (x_0,x_1,\dots),\, x_i\in X_A,\, \sigma_A(x_i) = x_{i-1},\, i = 1,2,\dots\big\}.$$

It follows from (76) that the condition

$$\sigma_A(x) = y, \qquad x,y\in X_A,$$

just means that

$$x = (\xi_0,\xi_1,\dots), \qquad y = (\xi_1,\xi_2,\dots),$$

where $A(\xi_k,\xi_{k+1}) = 1$, $k\in\mathbb{N}$. Hence for each element

$$\widetilde{x} = (x_0,x_1,\dots)\in\widetilde{X}$$

there is a unique sequence

$$(\xi_k)_{k\in\mathbb{Z}}$$

such that $x_i = (\xi_k)_{k=-i,\dots,+\infty}$, $i = 0,1,\dots$, and $A(\xi_k,\xi_{k+1}) = 1$, $k\in\mathbb{Z}$, that is, $\widetilde{X}\subset\overline{X}_A$. It is also clear that each sequence $(\xi_k)_{k\in\mathbb{Z}}\in\overline{X}_A$ determines a unique sequence $\widetilde{x} = (x_0,x_1,\dots)\in\widetilde{X}$ by the rule described above. Thus, $\widetilde{X} = \overline{X}_A$.

By (66), in our case the map $\widetilde{\sigma}\colon\widetilde{X}\to\widetilde{X}$ takes the following form:

$$\widetilde{\sigma}(\widetilde{x}) = (\sigma_A(x_0),x_0,x_1,\dots), \qquad \widetilde{x} = (x_0,x_1,\dots). \tag{78}$$

In view of the above identification of $\widetilde{X}$ and $\overline{X}_A$, formula (78) becomes

$$\widetilde{\sigma}(x)_k = \xi_{k+1}, \qquad k\in\mathbb{Z}, \quad x\in\overline{X}_A,$$

so it coincides with (77). The proof is complete.

Remark 5.3. A description of reversible extensions of topological Markov chains defined by matrices which are allowed to contain zero columns was presented in [9].

Example 5.4 (the Smale horseshoe and the maximal ideal space of the algebra corresponding to a reversible extension of a topological Markov chain). In the previous example we described a reversible extension of a topological Markov chain (which coincided in a natural fashion with a reversible topological Markov chain). This description was not explicitly geometrical. We can ask if the topological spaces and dynamical systems arising in this extension can be described in 'more explicit' geometrical terms. We shall show by this example that these objects are closely related with maps of the Smale horseshoe kind. We recall this construction, which was put forward by Smale in 1965, giving the first example of a structurally stable diffeomorphism (the 'horseshoe map') with an infinite set of periodic points (see, for instance, [10], Ch. 4).

The *horseshoe map* h stretches and bends the unit square S as follows into a figure similar to a horseshoe. First, S is uniformly contracted in the vertical direction with coefficient $\eta < 1/2$, then it is stretched in the horizontal direction with coefficient $1/\eta > 2$, and finally it is bent so that the intersection of the horseshoe $h(S)$ and S consists of two rectangles, as displayed in Fig. 1.

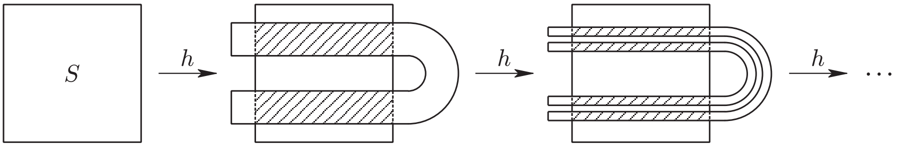

Figure 1. The images of the unit square S under the iterations of the horseshoe map h.

The set $h^n(S)\cap S$ consists of 2^n rectangles of height η^n (see Fig. 1), therefore $H^+ := \bigcap_{n=0}^{\infty} h^n(S)$ is the product of the (horizontal) unit interval by a (vertical) Cantor-type set. In a similar way, using inverse images (Fig. 2), we see that the set $H^- := \bigcap_{n=1}^{\infty} h^{-n}(S)$ is the product of a (vertical) unit interval by a (horizontal) Cantor-type set.

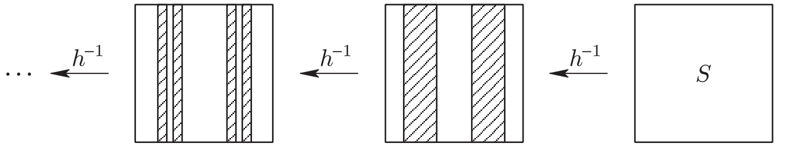

Figure 2. The inverse images of the unit square S under the iterations of the horseshoe map h.

The *horseshoe set* $H := H^+ \cap H^- = \bigcap_{n\in\mathbb{Z}} h^n(S)$ is a Cantor-type set. The horseshoe map $h\colon H \to H$ is a homeomorphism; moreover, we can encode points in H by two-sided sequences of zeros and ones so that h becomes the two-sided shift (Fig. 3).

In other words, the 'horseshoe' dynamical system (H,h) and the invertible topological Markov chain $(\overline{X}_A,\overline{\sigma}_A)$, where $A=\left(\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right)$, are topologically conjugate.

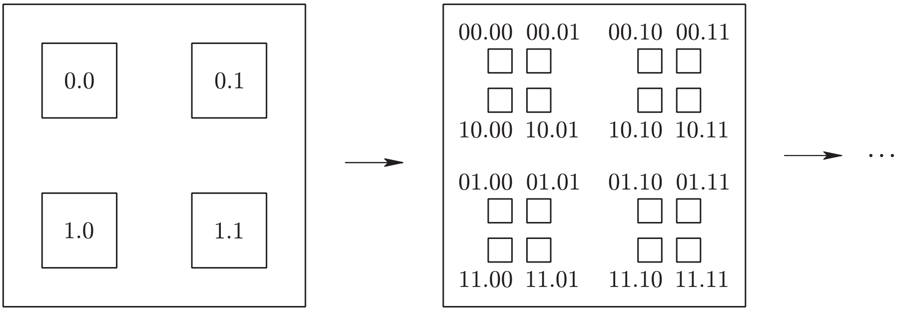

Figure 3. Encoding points in the horseshoe set H.

In this way we have obtained a geometrical illustration of Proposition 5.2. In particular, if $\mathscr{C}$ is the standard Cantor set on the unit interval and

$$f(x) := 3x \pmod 1, \qquad x \in \mathscr{C},$$

then the dynamical system $(\mathscr{C}, f)$ is equivalent to the one-sided topological Markov chain (X_A, σ_A) with matrix A as above; therefore, recalling Theorem 4.1 we conclude that *the horseshoe dynamical system* (H, h) *is a reversible extension of the dynamical system* $(\mathscr{C}, f)$, *and as regards the horseshoe set* H, *it is the maximal ideal space of the algebra corresponding to the reversible extension of the corresponding* C^**-dynamical system.*

Remark 5.5. Different dynamical systems may involve different types of horseshoe maps (which in our opinion could naturally be called *maps of coil-pipe type*). Using the above procedure we can obtain various descriptions of reversible extensions of topological Markov chains and the corresponding C^*-dynamical systems (Fig. 4; see [10]).

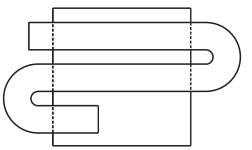

Figure 4. The image of the unit square in the geometrical representation of the reversible topological Markov chain $(\overline{X}_A, \overline{\sigma}_A)$ defined by the matrix $A = \left(\begin{smallmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{smallmatrix}\right)$.

Example 5.6 (the n-adic solenoid). We finish by giving examples of dynamical systems leading to n-adic solenoids.

We recall how these objects are constructed. Fix $n = 1, 2, 3, \dots$. Let

$$S^1 = \{z \in \mathbb{C} : |z| = 1\}$$

be the unit circle and

$$D^2 = \{z \in \mathbb{C} : |z| \leqslant 1\}$$

be the unit disc. By definition the *n-adic solenoid* $\mathscr{S}_n$ is the attractor of the map F_n acting on the solid torus $\mathscr{T} = S^1 \times D^2$ by the formula

$$F_n(z_1, z_2) = \left(z_1^n,\ \lambda z_2 + \frac{n-1}{n}\, z_1\right),$$

where $0 < \lambda < 1/n$ is fixed. In effect, the image of F_n is a solid torus wound n times round itself and inscribed in $\mathscr{T}$. It follows from this definition that the n-adic solenoid is the set

$$\mathscr{S}_n = \bigcap_{k\in\mathbb{N}} F_n^k(\mathscr{T}).$$

Locally, the solenoid is the product of an interval by a Cantor-type set. The solenoid map $F_n\colon \mathscr{S}_n \to \mathscr{S}_n$ is a homeomorphism. Let $p\colon \mathscr{T} \to S^1$. $p(z_1, z_2) = z_1$, be the projection onto the first coordinate. It is known (see, for instance, [11], Section 1. 9) that the map

$$\mathscr{S}_n \ni s \longmapsto \bigl(p(s), p(F^{-1}(s)), \dots, p(F^{-k}(s)), \dots\bigr) \in \prod_{k\in\mathbb{N}} S^1$$

establishes a topological conjugacy between $(\mathscr{S}_n, F_n)$ and the dynamical system formed by the inverse limit $\varprojlim(S^1, g_n)$ with respect to the winding map $g_n(z) = z^n$, $z \in S^1$, and the corresponding induced homeomorphism. In other words, by Theorems 3.5 and 4.1, *the n-adic solenoid $\mathscr{S}_n$ and the map F_n form a reversible extension of the dynamical system (S^1, g_n), where $g_n(z) = z^n$, $z \in S^1$; and the solenoid itself, $\mathscr{S}_n$, is the maximal ideal space of the algebra corresponding to the reversible extension of the corresponding C^*-dynamical system.*

To complete the discussion of this example we write down a C^*-dynamical system whose extension gives us the solenoid explicitly.

Let $H = L_2(\mathbb{R})$, and let $\mathscr{A} \subset L(H)$ be the algebra of operators of multiplication by periodic functions with period 2π. Identifying in a natural way the space $\mathbb{R} \pmod{2\pi}$ with the unit circle $C(S^1)$ we obtain an isomorphism $\mathscr{A} \cong C(S^1)$. Consider the unitary operator $U \in L(H)$ such that

$$(Uf)(x) = \sqrt{2}\, f(2x), \qquad (U^*f)(x) = \frac{1}{\sqrt{2}}\, f\left(\frac{x}{2}\right).$$

For $a \in \mathscr{A}$, UaU^* is the operator of multiplication by $a(2x)$ and U^*aU is the operator of multiplication by $a(x/2)$. Thus, $U\mathscr{A}U^* \subset \mathscr{A}$, $U^*\mathscr{A}U \not\subseteq \mathscr{A}$, and using complex notation for the variable on S^1 we obtain

$$UaU^*(z) = a(z^2), \qquad a \in \mathscr{A}.$$

In combination with the arguments presented above and Theorem 4.1, this formula leads to the following result describing the reversible extension of the C^*-dynamical system $(\mathscr{A}, U)$.

Proposition 5.7. *Let $(\mathscr{A}, U)$ be the C^*-dynamical system introduced above and let $(\overline{E_*(\mathscr{A})}, U)$ be its reversible extension described in Theorem 4.1. Then*

$$M(\overline{E_*(\mathscr{A})}) \cong C(\mathscr{S}_2)$$

and the automorphism $U(\,\cdot\,)U^*$ on the algebra $\overline{E_*(\mathscr{A})}$ is induced by the solenoid map F_2, that is,

$$UbU^* = b \circ F_2, \qquad b \in \overline{E_*(\mathscr{A})}.$$

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